15

CHAPTER

The Calculus of Residues (Integration)

15.1 ZERO OF ANALYTIC FUNCTION

A zero of analytic function $f(z)$ is the value of z for which $f(z) = 0$.

Example 1. *Find out the zeros and discuss the nature of the singularities of*

$$f(z) = \frac{(z-2)}{z^2} \sin\left(\frac{1}{z-1}\right)$$

Solution. Poles of $f(z)$ are given by equating to zero the denominator of $f(z)$ i.e. $z = 0$ is a pole of order two.

zeros of $f(z)$ are given by equating to zero the numerator of $f(z)$ *i.e.*, $(z-2)\ \sin\left(\frac{1}{z-1}\right) = 0$

$$\Rightarrow \qquad \text{Either } z - 2 = 0 \quad \text{or } \sin\left(\frac{1}{z-1}\right) = 0$$

$$\Rightarrow \qquad z = 2 \qquad \text{and} \quad \frac{1}{z-1} = n\pi$$

$$\Rightarrow \qquad z = 2, \qquad z = \frac{1}{n\pi} + 1, \quad n = \pm 1, \pm 2, \ldots\ldots\ldots$$

Thus, $z = 2$ is a simple zero. The limit point of the zeros are given by

$$z = \frac{1}{n\pi} + 1 \quad (n = \pm 1, \pm 2, \ldots\ldots\ldots\ldots) \text{ is } z = 1.$$

Hence $z = 1$ is an isolated essential singularity. **Ans.**

15.2 SINGULAR POINT

A point at which a function $f(z)$ is not analytic is known as a singular point or **singularity** of the function.

For example, the function $\frac{1}{z-2}$ has a singular point at $z - 2 = 0$ or $z = 2$.

Isolated singular point. If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called non-isolated.

For example, the function $\frac{1}{(z-1)(z-3)}$ has two isolated singular points, namely $z = 1$ and $z = 3$. $[(z-1)(z-3) = 0$ or $z = 1, 3]$.

Example of non-isolated singularity. Function $\frac{1}{\sin\frac{\pi}{z}}$ is not analytic at the points where $\sin\frac{\pi}{z} = 0$, *i.e.*, at the points $\frac{\pi}{z} = n\pi$ *i.e.*, the points $z = \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$. Thus $z = 1, \frac{1}{2}, \frac{1}{3}, \ldots, z = 0$ are the points of singularity. $z = 0$ is the **non-isolated singularity** of the function $\frac{1}{\sin\frac{\pi}{z}}$ because in the neighbourhood of $z = 0$, there are infinite number of other singularities $z = \frac{1}{n}$, where n is very large.

Pole of order *m*. Let a function $f(z)$ have an isolated singular point $z = a$, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

$$+ \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m} + \frac{b_{m+1}}{(z-a)^{m+1}} + \frac{b_{m+2}}{(z-a)^{m+2}} + \ldots \qquad \ldots (1)$$

In some cases it may happen that the coefficients $b_{m+1} = b_{m+2} = b_{m+3} = 0$, then (1) reduces to

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m}$$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots + \frac{1}{(z-a)^m}\{b_1(z-a)^{m-1} + b_2(z-a)^{m-2}$$

$$+ b_3(z-a)^{m-3} + \ldots\ldots + b_m\}$$

then $z = a$ is said to be a **pole of order *m*** of the function $f(z)$, when $m = 1$, the pole is said to be **simple pole**. In this case

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{z-a}$$

If the number of the terms of negative powers in expansion (1) is infinite, then $z = a$ is called an essential singular point of $f(z)$.

Example 2. *Define the singularity of a function. Find the singularity (ties) of the functions*

$$(i)\ f(z) = \sin\frac{1}{z} \qquad (ii)\ g(z) = \frac{e^{\frac{1}{z}}}{z^2}$$

(*U.P. III Semester, 2009-2010*)

Solution. See Art. 15.2 on page 486 for definition.

(*i*) We know that

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} + \ldots\ldots\ldots + (-1)^n \frac{1}{(2n+1)!\,z^{2n+1}}$$

Obviously, there is a number of singularity.

$$\sin\frac{1}{z} \text{ is not analytic at } z = 0. \qquad \left(\frac{1}{z} = \infty \text{ at } z = 0\right)$$

Hence, $\sin\frac{1}{z}$ has a singularity at $z = 0$.

(ii) Here, we have $g(z) = \dfrac{e^{\frac{1}{z}}}{z^2}$

We know that, $\left(\dfrac{1}{z^2}\right)\left(e^{\frac{1}{z}}\right) = \dfrac{1}{z^2}\left(1 + \dfrac{1}{z} + \dfrac{1}{2!z^2} + \dfrac{1}{3!z^3} + ... + \dfrac{1}{n!z^n} .... + \right)$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2!z^4} + \frac{1}{3!z^5} + ... \frac{1}{n!z^{n+2}} + ...$$

Here, $f(z)$ has infinite number of terms in negative powers of z.

Hence, $f(z)$ has essential singularity at $z = 0$. **Ans.**

Example 3. *Find the pole of the function* $\dfrac{e^{z-a}}{(z-a)^2}$

Solution. $\dfrac{e^{z-a}}{(z-a)^2} = \dfrac{1}{(z-a)^2}\left[1 + (z-a) + \dfrac{(z-a)^2}{2!} + ...\right]$

The given function has negative power 2 of $(z - a)$.

So, the given function has a pole at $z = a$ of order 2. **Ans.**

Example 4. *Find the poles of* $f(z) = \sin\left(\dfrac{1}{z-a}\right)$

Solution. $\sin\left(\dfrac{1}{z-a}\right) = \dfrac{1}{z-a} - \dfrac{1}{3!}\dfrac{1}{(z-a)^3} + \dfrac{1}{5!}\dfrac{1}{(z-a)^5} - .....$

The given function $f(z)$ has infinite number of terms in the negative powers of $z - a$.

So, $f(z)$ has essential singularity at $z = a$. **Ans.**

Example 5. *Find the pole of* $f(z) = \dfrac{\sin(z-a)}{(z-a)^4}$

Solution. $\dfrac{\sin(z-a)}{(z-a)^4} = \dfrac{1}{(z-a)^4}\left[(z-a) - \dfrac{(z-a)^3}{3!} + \dfrac{(z-a)^5}{5!} - \dfrac{(z-a)^7}{7!} + ...\right]$

$$= \frac{1}{(z-a)^3}\left[1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \frac{(z-a)^6}{7!} + ...\right]$$

The given function has a negative power 3 of $(z - a)$.

So, $f(z)$ has a pole at $z = a$ of order 3. **Ans.**

Example 6. *Prove that* $f(z) = \lim\limits_{z \to a} e^{\frac{1}{z-a}}$ *does not exist.*

Solution. $\lim\limits_{z \to a} e^{\frac{1}{z-a}} = \lim\limits_{z \to a}\left(1 + \dfrac{1}{z-a} + \dfrac{1}{2!(z-a)^2} + \dfrac{1}{3!(z-a)^3} + ... + \dfrac{1}{n!(z-a)^n} + ... \infty\right)$

Here $n \to \infty$, $f(z)$ has infinite number of terms in negative power of $(z - a)$.

Thus, $f(z)$ has essential singularity at $z = a$.

Hence, $f(z) = \lim\limits_{z \to a} e^{\frac{1}{z-a}}$ does not exist. **Ans.**

Example 7. *Discuss singularity of* $\dfrac{1}{1-e^z}$ *at* $z = 2\pi i$.

Solution. We have, $f(z) = \dfrac{1}{1-e^z}$

The poles are determined by putting the denominator equal to zero.

i.e., $1 - e^z = 0$

$\Rightarrow$ $e^z = 1 = (\cos 2n\pi + i\sin 2n\pi) = e^{2n\pi i}$

$\Rightarrow$ $z = 2n\pi i \quad (n = 0, \pm 1, \pm 2, ...)$

Clearly $z = 2\pi i$ is a simple pole. **Ans.**

Example 8. *Discuss singularity of* $\dfrac{\cot \pi z}{(z-a)^2}$ *at* $z = a$ *and* $z = \infty$.

Solution. Let $f(z) = \dfrac{\cot \pi z}{(z-a)^2} = \dfrac{\cos \pi z}{\sin \pi z (z-a)^2}$

The poles are given by putting the denominator equal to zero.

i.e., $\sin \pi z (z-a)^2 = 0 \Rightarrow (z-a)^2 = 0$ or $\sin \pi z = 0 = \sin n\pi$

$\Rightarrow$ $z = a, \ \pi z = n\pi, \ (n \in I)$

$\Rightarrow$ $z = a, n$

$f(z)$ has essential singularity at $z = \infty$.

Also, $z = a$ being repeated twice gives the double pole. **Ans.**

Example 9. *Show that* $e^{-\left(\frac{1}{z^2}\right)}$ *has no singularities.*

Solution. $f(z) = e^{-\left(\frac{1}{z^2}\right)} = \dfrac{1}{e^{(1/z^2)}}$

The poles are determined by putting the denominator

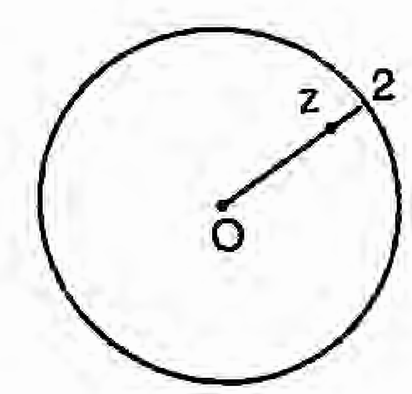
$$e^{\left(\frac{1}{z^2}\right)} = 0 \qquad ...(1)$$

It is not possible to find the value of z which can satisfy equation (1).

Hence, there is no pole or singularity of the given function. **Proved.**

Example 10. *Define the Laurent series expansion of a function Expand* $f(z) = e^{\frac{z}{(z-2)}}$ *in a Laurent series about the point* $z = 2$. *(U.P., III Semester, Dec. 2009)*

Solution. Here, we have



$$f(z) = e^{\frac{z}{z-2}}$$

$$= e^{\frac{z-2+2}{z-2}}$$

$$= e \cdot e^{\frac{2}{z-2}}$$

$$= e\, e^{\frac{2}{-2\left(1-\frac{z}{2}\right)}}$$

$$= e^1 . e^{-\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+....\right]}$$ [Binomial Theorem]

$$= e^{(1-1)} e^{-\left[\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}....\right]}$$

$$= e^{-\left[\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}.....\right]}$$

$$\left[e^z = 1+\frac{z}{1!}+\frac{z^2}{2!}+.......\right]$$

$$= \left[1-\frac{z}{2}-\frac{z^2}{4}-\frac{z^3}{8}+\frac{\left(\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}\right)^2}{2!}-\frac{\left(\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}\right)^3}{3!}+....\right]$$

$$= \left[1-\frac{z}{2}-\frac{z^2}{4}-\frac{z^3}{8}+\frac{z^2}{8}+\frac{z^4}{32}+\frac{z^3}{8}+\frac{z^4}{16}-\frac{z^3}{48}.........\right]$$

$$= 1-\frac{z}{2}-\frac{z^2}{8}-\frac{z^3}{48}.........$$

Ans.

Example 11. *Find the nature of singularities of*

$$f(z) = \frac{z-\sin z}{z^3} \quad at \quad z = 0.$$

Solution. $f(z) = \frac{1}{z^3}(z-\sin z)$ $= \frac{1}{z^3}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+.....\right)\right]$

$$= \frac{1}{z^3}\left(\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}-.....\right) = \frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-......$$

There is no negative power of z;

Hence, there is no pole. **Ans.**

Example 12. *Determine the poles of the function z*

$$f(z) = \frac{1}{z^4+1}$$

Solution.

$$f(z) = \frac{1}{z^4+1}$$

The poles of $f(z)$ are determined by putting the denominator equal to zero.

i.e., $z^4+1 = 0 \quad \Rightarrow \quad z^4 = -1$

$$z = (-1)^{\frac{1}{4}} = (\cos\pi + i\sin\pi)^{\frac{1}{4}}$$

$$= [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{\frac{1}{4}}$$ [By De Moiver's theorem]

$$= \left[\cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right]$$

If $n = 0$, Pole at $z = \left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right] = \left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$

If $n = 1$, Pole at $z = \left[\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right] = \left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$

If $n = 2$, Pole at $z = \left[\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right] = \left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$

If $n = 3$, Pole at $z = \left[\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right] = \left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$ **Ans.**

Example 13. *Show that the function e^z has an isolated essential singularity at $z = \infty$.*

Solution. Let $f(z) = e^z$

Putting $z = \frac{1}{t}$, we get

$$f\left(\frac{1}{t}\right) = e^{\frac{1}{t}} = 1+\frac{1}{t}+\frac{1}{2!.t^2}+\frac{1}{3!\,t^3}+.....$$

Here, the principal part of $f\left(\frac{1}{t}\right)$;

$$\frac{1}{t}+\frac{1}{2!\,t^2}+\frac{1}{3!\,t^3}+...$$

Contains infinite number of terms.

Hence $t = 0$ is an isolated essential singularity of $e^{\frac{1}{t}}$ and $z = \infty$ is an isolated essential singularity of e^z. **Ans.**

EXERCISE 15.1

Find the poles or singularity of the following functions:

1. $\frac{1}{(z-2)(z-3)}$ **Ans.** 2 simple poles at $z = 2$ and $z = 3$.
2. $\frac{e^z}{(z-2)^3}$ **Ans.** Pole at $z = 2$ of order 3.
3. $\frac{1}{\sin z - \cos z}$ **Ans.** Simple pole at $z = \frac{\pi}{4}$
4. $\cot\frac{1}{z}$ **Ans.** Essential singularity at $z = 0$
5. $z \operatorname{cosec} z$ **Ans.** Non-isolated essential singularity
6. $\sin\frac{1}{z}$ **Ans.** Essential singularity

Choose the correct alternative:

7. Let $f(z) = \dfrac{1}{(z-2)^4 (z+3)^6}$, then $z = 2$ and $z = -3$ are the poles of order :

(*R.G.P.V., Bhopal III Semester, June 2007*)

(*a*) 6 and 4 (*b*) 2 and 3 (*c*) 3 and 4 (*d*) 4 and 6 **Ans.** (*d*)

15.3 THEOREM

If $f(z)$ has a pole at $z = a$, then $|f(z)| \to \infty$ as $z \to a$.

Proof. Let $z = a$ be a pole of order m of $f(z)$. Then by Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{m=1}^{m} b_n (z-a)^{-n}$$

$$= \sum_{0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots\ldots + \frac{b_m}{(z-a)^m}$$

$$= \sum_{0}^{\infty} a_n (z-a)^n + \frac{1}{(z-a)^m} [b_1 (z-a)^{m-1} + b_2 (z-a)^{m-2} + \ldots\ldots + b_{m-1}(z-a) + b_m]$$

$$= \sum_{0}^{\infty} a_n (z-a)^n + \frac{\varphi(z)}{(z-a)^m}$$

Now $\varphi(z) \to b_m$ as $z \to a$.

Hence $|f(z)| \to \infty$ as $z \to a$. **Proved.**

Example 14. *If an analytic function $f(z)$ has a pole of order m at $z = a$, then $\dfrac{1}{f(z)}$ has a zero of order m at $z = a$.*

Solution. If $f(z)$ has a pole of order m at $z = a$, then

$$f(z) = \frac{\varphi(z)}{(z-a)^m} \quad \text{where } \varphi(z) \text{ is analytic and non-zero at } z = a.$$

$$\therefore \quad \frac{1}{f(z)} = \frac{(z-a)^m}{\varphi(z)}$$

Clearly, $\dfrac{1}{f(z)}$ has a zero of order m at $z = a$, since $\phi(a) \neq 0$.

15.4 DEFINITION OF THE RESIDUE AT A POLE

Let $z = a$ be a pole of order m of a function $f(z)$ and C_1 circle of radius r with centre at $z = a$ which does not contain any other singularities except at $z = a$ then $f(z)$ is analytic within the annulus $r < |z-a| < R$ and can be expanded within the annulus. Laurent's series:

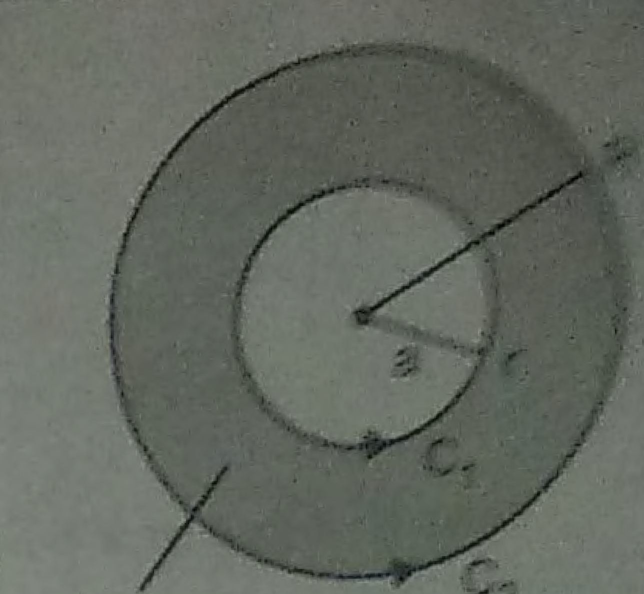
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad \ldots(1)$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^{n+1}} \quad \ldots(2)$$

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{-n+1}}\, dz \quad \ldots(3)$$

$|z-a| = r$ being the circle C_1.



Annulus

Particularly,
$$b_1 = \frac{1}{2\pi i} \int_{C_1} f(z)\, dz$$

The coefficient b_1 is called residue of $f(z)$ at the pole $z = a$. It is denoted by symbol

$$\text{Res. } (z = a) = b_1.$$

15.5 RESIDUE AT INFINITY

Residue of $f(z)$ at $z = \infty$ is defined as $-\dfrac{1}{2\pi i}\int_C f(z)\, dz$ where the integration is taken round C in anti-clockwise direction, where C is a large circle containing all finite singularities of $f(z)$.

15.6 METHOD OF FINDING RESIDUES

(*a*) **Residue at simple pole**

(*i*) If $f(z)$ has a simple pole at $z = a$, then

$$\text{Res } f(a) = \lim_{z \to a} (z-a) f(z)$$

Proof.
$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{z-a}$$

$$\Rightarrow \quad (z-a) f(z) = a_0(z-a) + a_1(z-a)^2 + a_2(z-a)^3 + \ldots + b_1$$

$$\Rightarrow \quad b_1 = (z-a) f(z) - [a_0(z-a) + a_1(z-a)^2 + a_2(z-a)^3 + \ldots]$$

Taking limit as $z \to a$, we have $b_1 = \lim_{z \to a} (z-a) f(z)$

$$\boxed{\text{Res } (at\ z = a) = \lim_{z \to a} (z-a) f(z)}$$ **Proved.**

(*ii*) If $f(z)$ is of the form $f(z) = \dfrac{\phi(z)}{\psi(z)}$ where $\psi(a) = 0$, but $\phi(a) \neq 0$

$$\text{Res } (at\ z = a) = \frac{\phi(a)}{\psi'(a)}$$

Proof.
$$f(z) = \frac{\phi(z)}{\psi(z)}$$

$$\text{Res (at } z = a) = \lim_{z \to a} (z-a) f(z) = \lim_{z \to a} (z-a) \frac{\phi(z)}{\psi(z)}$$

$$= \lim_{z \to a} \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \ldots]}{\psi(a) + (z-a)\psi'(a) + \frac{(z-a)^2}{2!}\psi''(a) + \ldots} \quad \text{(By Taylor's Theorem)}$$

$$= \lim_{z \to a} \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \ldots]}{(z-a)\psi'(a) + \frac{(z-a)^2}{2!}\psi''(a) + \ldots} \quad [\text{since } \psi(a) = 0]$$

$$= \lim_{z \to a} \frac{\phi(a) + (z-a)\,\phi'(a) + \ldots}{\psi'(a) + \frac{z-a}{2!}\psi''(a) + \ldots}$$

$$\textbf{Res (at } z = a\textbf{)} = \frac{\phi(a)}{\psi'(a)}$$ **Proved.**

(*b*) **Residue at a pole of order *n*.** If $f(z)$ has a pole of order n at $z = a$, then

$$\textbf{Res (at } z = a\textbf{)} = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

Proof. If $z = a$ is a pole of order n of function $f(z)$, then by Laurent's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_n}{(z-a)^n}$$

Multiplying by $(z-a)^n$, we get

$$(z-a)^n f(z) = a_0(z-a)^n + a_1(z-a)^{n+1} + a_2(z-a)^{n+2} + \ldots$$
$$+\, b_1(z-a)^{n-1} + b_2(z-a)^{n-2} + b_3(z-a)^{n-3} + \ldots + b_n$$

Differentiating both sides w.r.t. 'z' $(n-1)$ times and putting $z = a$, we get

$$\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a} = (n-1)!\,b_1 \quad \Rightarrow \quad b_1 = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

$$\textbf{Residue } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

(*c*) **Residue at a pole $z = a$ of any order (simple or of order *m*)**

$$\textbf{Res } f(a) = \textbf{coefficient of } \frac{1}{t}$$

Proof. If $f(z)$ has a pole of order m, then by Laurent's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 \ldots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m}$$

If we put $z - a = t$ or $z = a + t$, then

$$f(a+t) = a_0 + a_1 t + a_2 t^2 + \ldots + \frac{b_1}{t} + \frac{b_2}{t^2} + \ldots + \frac{b_m}{t^m}$$

Res $f(a) = b_1$, **Res $f(a)$ = coefficient of $\frac{1}{t}$**

Rule. Put $z = a + t$ in the function $f(z)$, expand it in powers of t. Coefficient of $\frac{1}{t}$ is the residue of $f(z)$ at $z = a$.

(*d*) **Residue of $f(z)$ at $z = \infty$**

$$= \lim_{z \to \infty} \{-z\, f(z)\}$$

or The residue of $f(z)$ at infinity $= -\frac{1}{2\pi i}\int_c f(z)\,dz$

15.7 RESIDUE BY DEFINITION

Example 15. *Find the residue at $z = 0$ of $z \cos \frac{1}{z}$.*

Solution. Expanding the function in powers of $\frac{1}{z}$, we have

$$z \cos\frac{1}{z} = z\left[1 - \frac{1}{2z^2} + \frac{1}{4!\,z^4} - \ldots\right] = z - \frac{1}{2z} + \frac{1}{24z^3} - \ldots$$

This is the Laurent's expansion about $z = 0$.

The coefficient of $\frac{1}{z}$ in it is $-\frac{1}{2}$. So the residue of $z\cos\frac{1}{z}$ at $z = 0$ is $-\frac{1}{2}$. **Ans.**

Example 16. *Find the residue of $f(z) = \frac{z^3}{z^2 - 1}$ at $z = \infty$.*

Solution. We have, $f(z) = \frac{z^3}{z^2-1}$

$$f(z) = \frac{z^3}{z^2\left(1 - \frac{1}{z^2}\right)} = z\left(1 - \frac{1}{z^2}\right)^{-1}$$

$$= z\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \ldots\right) = z + \frac{1}{z} + \frac{1}{z^3} + \ldots$$

$$\text{Residue at infinity} = -\left(\text{coeff. of } \frac{1}{z}\right) = -1.$$ **Ans.**

15.8 FORMULA: RESIDUE $= \lim_{z \to a} (z-a)\, f(z)$

Example 17. *Determine the pole and residue at the pole of the function $f(z) = \frac{z}{z-1}$*

Solution. The poles of $f(z)$ are given by putting the denominator equal to zero.

$\therefore$ $z - 1 = 0 \Rightarrow z = 1$

The function $f(z)$ has a simple pole at $z = 1$.

Residue is calculated by the formula

$$\text{Residue} = \lim_{z \to a} (z-a)\, f(z)$$

$$\text{Residue of } f(z) \text{ at (at } z = 1) = \lim_{z \to 1} (z-1)\left(\frac{z}{z-1}\right)$$

$$= \lim_{z \to 1} (z) = 1$$

Hence, $f(z)$ has a simple pole at $z = 1$ and residue at the pole is 1. **Ans.**

Example 18. *Determine the poles and the residue at simple pole of the function*

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

Solution. The pole of $f(z)$ are given by putting the denominator equal to zero.

$$(z-1)^2(z+2) = 0 \quad \Rightarrow \quad z = 1, 1, -2$$

The function $f(z)$ has simple pole at $z = -2$ and at $z = 1$ pole of second order.

Residue of $f(z)$ at $z = -2$ is $= \lim\limits_{z \to -2} (z+2) f(z)$ $\quad$ [Residue $= \lim\limits_{z \to a} (z-a) f(z)$]

$$= \lim_{z \to -2} (z+2) \frac{z^2}{(z-1)^2 (z+2)}$$

$$= \lim_{z \to -2} \frac{z^2}{(z-1)^2} = \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$$

Hence, residue at simple pole is $\frac{4}{9}$. **Ans.**

Example 19. *Find the order of each pole and residue at it of* $\frac{1-2z}{z(z-1)(z-2)}$.

Solution. Let $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

The poles of $f(z)$ are given by $z(z-1)(z-2) = 0$

$\Rightarrow$ $z = 0, 1, 2$ all are simple poles.

Residue of $f(z)$ at $(z = 0) = \lim\limits_{z \to 0} (z-0) f(z) = \lim\limits_{z \to 0} \frac{z(1-2z)}{z(z-1)(z-2)}$

$$= \lim_{z \to 0} \frac{1-2z}{(z-1)(z-2)} = \frac{1}{2}$$

Residue of $f(z)$ at $(z = 1) = \lim\limits_{z \to 1} (z-1) f(z) = \lim\limits_{z \to 1} \frac{(z-1)(1-2z)}{z(z-1)(z-2)} = \lim\limits_{z \to 1} \frac{1-2z}{z(z-2)} = 1$

Residue of $f(z)$ at $(z = 2) = \lim\limits_{z \to 2} (z-2) f(z) = \lim\limits_{z \to 2} \frac{(z-2)(1-2z)}{z(z-1)(z-2)}$

$$= \lim_{z \to 2} \frac{1-2z}{z(z-1)} = -\frac{3}{2}$$

Hence, the residues of $f(z)$ at $z = 0$, $z = 1$ and $z = 2$ are $\frac{1}{2}, 1$ and $-\frac{3}{2}$ respectively. **Ans.**

Example 20. *Determine the residue of* $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$ *at its simple poles.*

Solution. The poles of $f(z)$ are determined by putting the denominator equal to zero.

i.e. $\quad (z-1)^4 (z-2)(z-3) = 0$

$\Rightarrow \quad z = 1, 1, 1, 1$ and $z = 2$ and $z = 3$

The simple poles of the function $f(z)$ are at $z = 2$ and $z = 3$.

Pole at $z = 2$

Residue, $\quad R(2) = \lim\limits_{z \to 2} (z-2) \frac{z^3}{(z-1)^4 (z-2)(z-3)}$

[Residue $R(2) = \lim\limits_{z \to 2} [(z-2) f(z)]$]

$$= \lim_{z \to 2} \frac{z^3}{(z-1)^4 (z-3)} = \frac{(2)^3}{(1)^4 (-1)} = -8$$

Pole at $z = 3$

Residue, $\quad R(3) = \lim\limits_{z \to 3} (z-3) \frac{z^3}{(z-1)^4 (z-2)(z-3)}$

$$= \lim_{z \to 3} \frac{z^3}{(z-1)^4 (z-2)} = \frac{(3)^3}{(3-1)^4 (3-2)} = \frac{27}{16}$$

Hence, residue at $z = 2$ and $z = 3$ are -8 and $\frac{27}{16}$ respectively. **Ans.**

Example 21. *Evaluate the residues of* $\frac{z^2}{(z-1)(z-2)(z-3)}$ *at* $z = 1, 2, 3$ *at infinity and show that their sum is zero.*

Solution. Let $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$

The poles of $f(z)$ are determined by putting the denominator equal to zero.

$\therefore \quad (z-1)(z-2)(z-3) = 0$

$\Rightarrow \quad z = 1, 2, 3$

Residue of $f(z)$ at $(z = 1) = \lim\limits_{z \to 1} (z-1) f(z)$

$$= \lim_{z \to 1} (z-1) \cdot \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z \to 1} \frac{z^2}{(z-2)(z-3)} = \frac{1}{2}$$

Residue of $f(z)$ at $(z = 2) = \lim\limits_{z \to 2} (z-2) f(z)$

$$= \lim_{z \to 2} (z-2) \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z \to 2} \frac{z^2}{(z-1)(z-3)} = \frac{4}{(1)(-1)} = -4$$

Residue of $f(z)$ at $(z = 3) = \lim\limits_{z \to 3} (z-3) f(z)$

$$= \lim_{z \to 3} (z-3) \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z \to 3} \frac{z^2}{(z-1)(z-2)} = \frac{9}{2}$$

Residue of $f(z)$ at $(z = \infty) = \lim\limits_{z \to \infty} -z f(z) = \frac{-z(z^2)}{(z-1)(z-2)(z-3)}$

$$= \lim_{z \to \infty} \frac{-1}{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)} = -1$$

Sum of the residues at all the poles of $f(z)$

$$= \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0$$

Hence, the sum of the residues is zero.

Proved.

15.9 FORMULA: RESIDUE OF $f(\text{at } z = a) = \lim_{z \to a} \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$

Example 22. *Find the residue of a function*

$$f(z) = \frac{z^2}{(z+1)^2(z-2)} \text{ at its double pole.}$$

Solution. We have, $f(z) = \dfrac{z^2}{(z+1)^2(z-2)}$

Poles are determined by putting denominator equal to zero.

i.e.; $(z+1)^2(z-2) = 0$

$\Rightarrow$ $z = -1, -1$ and $z = 2$

The function has a double pole at $z = -1$

$$\text{Residue at } (z=-1) = \lim_{z \to -1} \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z+1)^2 \frac{z^2}{(z+1)^2(z-2)}\right\}\right]$$

$$= \left[\frac{d}{dz}\left(\frac{z^2}{z-2}\right)\right]_{z=-1}$$

$$= \left(\frac{(z-2)\,2z - z^2 \cdot 1}{(z-2)^2}\right)_{z=-1}$$

$$= \left[\frac{z^2-4z}{(z-2)^2}\right]_{z=-1} = \frac{(-1)^2-4(-1)}{(-1-2)^2}$$

$$\text{Residue at } (z=-1) = \frac{1+4}{9} = \frac{5}{9}$$

Ans.

Example 23. *Find the residue of* $\dfrac{1}{(z^2+1)^3}$ *at* $z = i$.

Solution. Let $f(z) = \dfrac{1}{(z^2+1)^3}$

The poles of $f(z)$ are determined by putting denominator equal to zero.

i.e.; $(z^2+1)^3 = 0$

$\Rightarrow$ $(z+i)^3(z-i)^3 = 0$

$\Rightarrow$ $z = \pm i$

Here, $z = i$ is a pole of order 3 of $f(z)$.

Residue at $z = i$:

$$= \lim_{z \to i} \frac{1}{(3-1)!}\left\{\frac{d^{3-1}}{dz^{3-1}}\left[(z-i)^3 \frac{1}{(z^2+1)^3}\right]\right\}$$

$$= \lim_{z \to i} \frac{1}{2!}\left\{\frac{d^2}{dz^2}\left(\frac{1}{(z+i)^3}\right)\right\}$$

$$= \lim_{z \to i} \frac{1}{2}\left(\frac{3 \times 4}{(z+i)^5}\right)$$

$$= \frac{1}{2} \times \frac{12}{(i+i)^5} = \frac{6}{32i} = \frac{3}{16i} = -\frac{3i}{16}$$

Hence, the residue of the given function at $z = i$ is $-\dfrac{3i}{16}$. **Ans.**

15.10 FORMULA: RES. (AT $z = a$) = $\dfrac{\phi(a)}{\psi'(a)}$

Example 24. *Determine the poles and residue at each pole of the function $f(z) = \cot z$.*

(AKTU, 2016, 2017)

Solution. $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

The poles of the function $f(z)$ are given by

$\sin z = 0$, $z = n\pi$, where $n = 0, \pm 1, \pm 2, \pm 3 \ldots$

$$\text{Residue of } f(z) \text{ at } z = n\pi \text{ is} = \frac{\cos z}{\frac{d}{dz}(\sin z)} = \frac{\cos z}{\cos z} = 1 \quad \left[\text{Res. at } (z=a) = \frac{\phi(a)}{\psi'(a)}\right]$$ **Ans.**

Example 25. *Determine the poles of the function and residue at the poles.*

$$f(z) = \frac{z}{\sin z}$$

Solution. $f(z) = \dfrac{z}{\sin z}$

Poles are determined by putting $\sin z = 0 = \sin n\pi \Rightarrow z = n\pi$

$$\text{Residue} = \left(\frac{z}{\cos z}\right)_{z=n\pi} \quad \left[\text{Residue} = \frac{\phi(a)}{\psi'(a)}\right]$$

$$= \frac{n\pi}{\cos n\pi}$$

$$= \frac{n\pi}{(-1)^n}$$

Hence, the residue of the given function at pole $z = n\pi$ is $\dfrac{n\pi}{(-1)^n}$ **Ans.**

FORMULA: RESIDUE = COEFFICIENT OF $\frac{1}{t}$

where $z = \frac{1}{t}$

Example 26. *Find the residue of* $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ *at a pole of order 4.*

Solution. The poles of $f(z)$ are determined by $(z-1)^4(z-2)(z-3) = 0 \Rightarrow z = 1, 2, 3$

Here $z = 1$ is a pole of order 4.

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)} \qquad ...(1)$$

Putting $z - 1 = t$ or $z = 1 + t$ in (1), we get

$$f(1+t) = \frac{(1+t)^3}{t^4(t-1)(t-2)}$$

$$= \frac{1}{t^4}(t^3+3t^2+3t+1)(1-t)^{-1}\frac{1}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(\frac{1}{t}+\frac{3}{t^2}+\frac{3}{t^3}+\frac{1}{t^4}\right)(1+t+t^2+t^3+...)\times\left(1+\frac{t}{2}+\frac{t^2}{4}+\frac{t^3}{8}...\right)$$

$$= \frac{1}{2}\left(\frac{1}{t}+\frac{3}{t^2}+\frac{3}{t^3}+\frac{1}{t^4}\right)\left(1+\frac{3}{2}t+\frac{7}{4}t^2+\frac{15}{8}t^3+...\right) = \frac{1}{2}\left(\frac{1}{t}+\frac{9}{2t}+\frac{21}{4}\frac{1}{t}+\frac{15}{8}\frac{1}{t}\right)+...$$

$$= \frac{1}{2}\left(1+\frac{9}{2}+\frac{21}{4}+\frac{15}{8}\right)\frac{1}{t} \qquad \left[\text{Res } f(a) = \text{coeffi. of } \frac{1}{t}\right]$$

Coefficient of $\frac{1}{t} = \frac{1}{2}\left(1+\frac{9}{2}+\frac{21}{4}+\frac{15}{8}\right) = \frac{101}{16}$,

Hence, the residue of the given function at a pole of order 4 is $\frac{101}{16}$. **Ans.**

Example 27. *Find the residue of* $f(z) = \frac{ze^z}{(z-a)^3}$ *at its pole.*

Solution. The pole of $f(z)$ is given by $(z-a)^3 = 0$ *i.e.*, $z = a$

Here $z = a$ is a pole of order 3.

Putting $z = a = t$ where t is small.

$$f(z) = \frac{ze^z}{(z-a)^3} \Rightarrow f(z) = \frac{(a+t)e^{a+t}}{t^3} = \left(\frac{a}{t^3}+\frac{1}{t^2}\right)e^{a+t} = e^a\left(\frac{a}{t^3}+\frac{1}{t^2}\right)e^t \qquad (z = a+t)$$

$$= e^a\left(\frac{a}{t^3}+\frac{1}{t^2}\right)\left(1+\frac{t}{1!}+\frac{t^2}{2!}+...\right) = e^a\left[\frac{a}{t^3}+\frac{a}{t^2}+\frac{a}{2t}+\frac{1}{t^2}+\frac{1}{t}+\frac{1}{2}+...\right]$$

$$= e^a\left[\frac{1}{2}+\left(\frac{a}{2}+1\right)\frac{1}{t}+(a+1)\frac{1}{t^2}+(a)\frac{1}{t^3}+...\right]$$

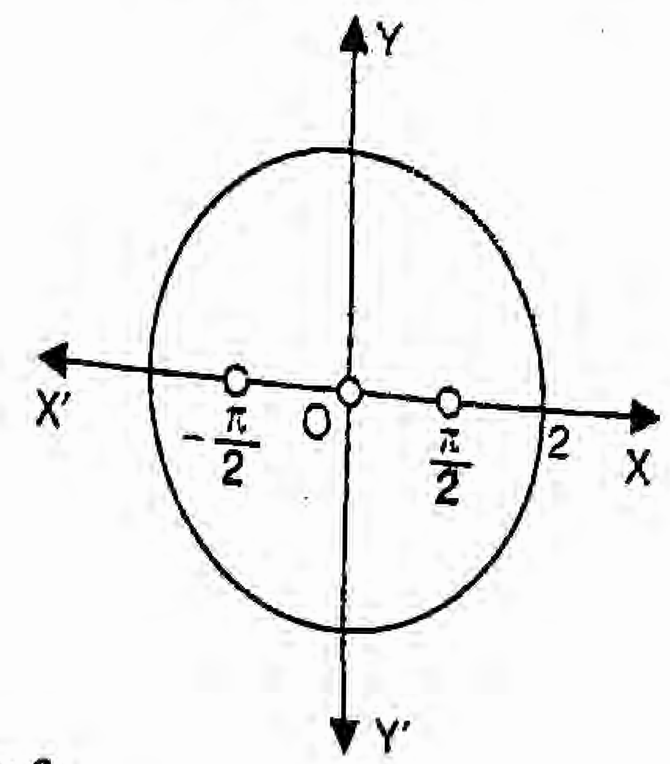
Coefficient of $\frac{1}{t} = e^a\left(\frac{a}{2}+1\right)$

Hence the residue at $z = a$ is $e^a\left(\frac{a}{2}+1\right)$ **Ans.**

Example 28. *Find the sum of the residues of the function* $f(z) = \frac{\sin z}{z\cos z}$ *at its poles inside the circle* $|z| = 2$.

Solution. We have,

$$f(z) = \frac{\sin z}{z\cos z}$$



The pole can be determined by putting denominator

$$z\cos z = 0$$

$$\Rightarrow \quad z = 0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$$

Of these poles only $z = 0$, $z = \pm\frac{\pi}{2}$ lie inside a circle $|z| = 2$.

Residue of $f(z)$ at $z = 0$ is $\lim_{z\to 0}|z \cdot f(z)| = \lim_{z\to 0}\frac{\sin z}{\cos z} = 0 \qquad ...(1)$

Residue of $f(z)$ at $z = \frac{\pi}{2}$ is

$$\lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)f(z) = \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)\sin z}{z\cos z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)\cos z+\sin z}{\cos z - z\sin z} \qquad \text{[By L' Hopital's Rule]}$$

$$= \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi} \qquad ...(2)$$

Similarly, residue of $f(z)$ at $z = -\frac{\pi}{2}$ is $\frac{2}{\pi} \qquad ...(3)$

$\therefore$ Sum of the residues $= 0 - \frac{2}{\pi} + \frac{2}{\pi} = 0.$ **Ans.**

EXERCISE 15.2

1. Determine the poles of the following functions. Find the order of each pole.

(i) $\frac{z^2}{(z-a)(z-b)(z-c)}$ **Ans.** Simple poles at $z = a$, $z = b$, $z = c$

(ii) $\frac{z-3}{(z-2)^2(z+1)}$ **Ans.** Pole at $z = 2$ of second order and $z = -1$ of first order.

(iii) $\frac{z e^{iz}}{z^2+a^2}$ **Ans.** Poles at $z = \pm\, ia$, order 1.

(iv) $\frac{1}{(z-1)(z-2)}$ **Ans.** $z = 2, z = 1$

Find the residue of

2. $\frac{z^3}{(z-2)(z-3)}$ at its poles. **Ans.** 27, – 8

3. $\frac{z^2}{z^2+a^2}$ at $z = ia$. **Ans.** $\frac{1}{2} ia$

4. $\frac{1}{(z^2+a^2)^2}$ at $z = ia$ **Ans.** $-\frac{i}{4a^3}$

5. $\tan z$ at its pole. **Ans.** –1 at its poles $f\left(n+\frac{\pi}{2}\right)$

6. $z^2 e^{1/z}$ at the point $z = 0$. **Ans.** $\frac{1}{6}$

7. $z^2 \sin\left(\frac{1}{z}\right)$ at $z = 0$ **Ans.** $-\frac{1}{6}$

8. $\frac{1}{z^2(z-i)}$ at $z = i$ **Ans.** –1

9. $\frac{e^{2z}}{1-e^z}$ at its pole **Ans.** – 1

10. $\frac{1+e^z}{\sin z + z\cos z}$ at $z = 0$ **Ans.** 1

11. $\frac{1}{z(e^z-1)}$ at its poles **Ans.** $-\frac{1}{2}$

Choose the correct answers:

12. The function $f(z) = \left\{\sin\left(\frac{1}{z}\right)\right\}^{-1}$ has multiple poles all of which are isolated singularity

(i) False (ii) True (iii) Partially true (iv) None of these

Ans. (i)

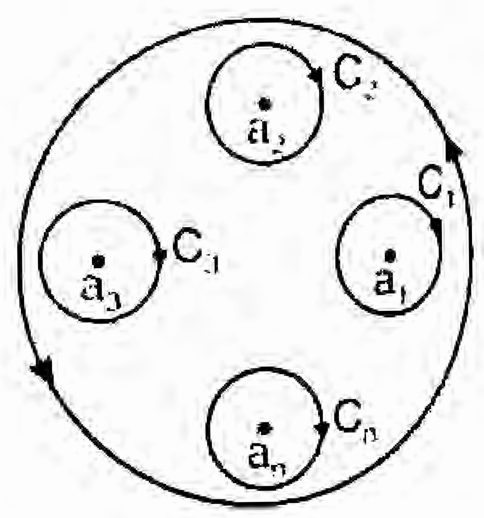
13. The residue of a function can be evaluated only if the pole is an isolated singularity.

(i) True (ii) False (iii) Partially false (iv) None of these

Ans. (i)

15.12 RESIDUE THEOREM

If $f(z)$ is analytic in a closed curve C, except at a finite number of poles within C, then $\int_C f(z)\,dz = 2\pi i$ **(sum of residues at the poles within C).**



Proof. Let $C_1, C_2, C_3, \ldots, C_n$ be the non-intersecting circles with centres at $a_1, a_2, a_3, \ldots, a_n$ respectively, and radii so small that they lie entirely within the closed curve C. Then $f(z)$ is analytic in the multiple connected region lying between the curves C and $C_1, C_2, C_3, \ldots, C_n$.

Applying ...

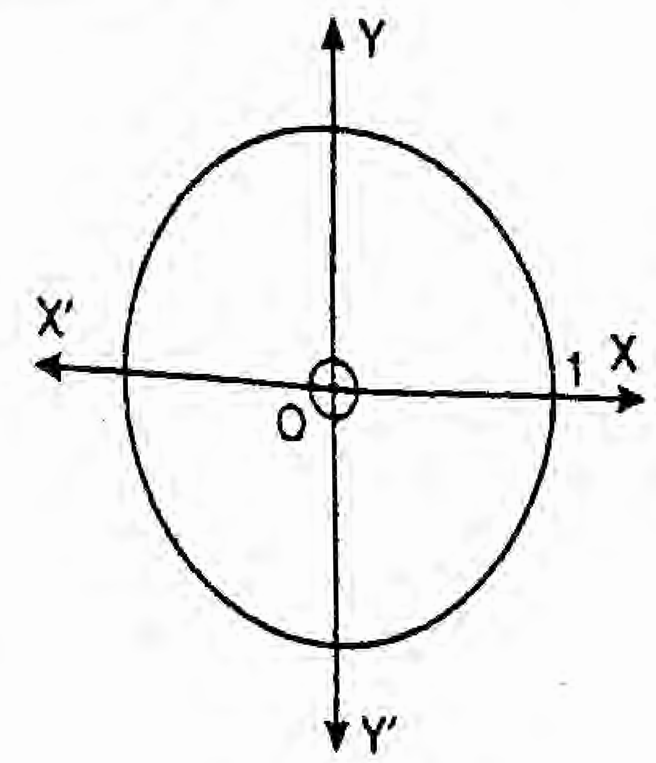
$$\int_C f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + \ldots + \int_{c_n} f(z)\,dz.$$

$$= 2\pi i\,[\text{Res } f(a_1) + \text{Res } f(a_2) + \text{Res } f(a_3) + \ldots + \text{Res } f(a_n)]$$ **Proved.**

Example 29. *Evaluate the following integral using residue theorem*

$$\int_c \frac{1+z}{z(2-z)}\,dz$$

where c is the circle $|z| = 1$.



Solution. The poles of the integrand are given by putting the denominator equal to zero.

$$z(2-z) = 0 \quad \text{or} \quad z = 0, 2$$

The integrand is analytic on $|z| = 1$ and all points inside except $z = 0$, as a pole at $z = 0$ is inside the circle $|z| = 1$.

Hence by residue theorem

$$\int_c \frac{1+z}{z(2-z)}\,dz = 2\pi i\ [\text{Res } f(0)] \quad \ldots (1)$$

$$\text{Residue } f(0) = \lim_{z\to 0} z.\frac{1+z}{z(2-z)} = \lim_{z\to 0}\frac{1+z}{2-z} = \frac{1}{2}$$

Putting the value of Residue $f(0)$ in (1), we get

$$\int_c \frac{1+z}{z(2-z)}\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i$$ **Ans.**

Example 30. *Evaluate the following integral using residue theorem*

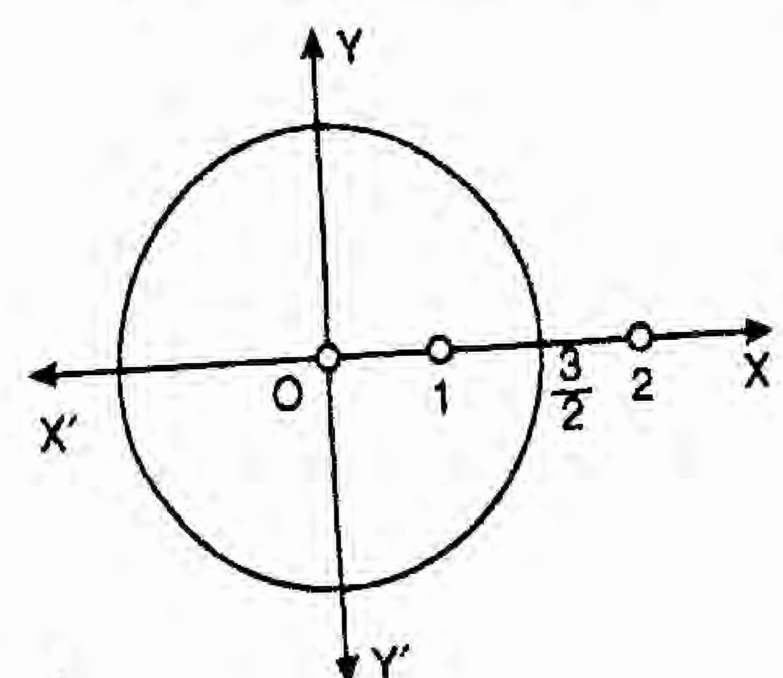
$$\int_c \frac{4-3z}{z(z-1)(z-2)}\,dz$$

where c is the circle $|z| = \frac{3}{2}$.

Solution. The poles of the function $f(z)$ are given by equating the denominator to zero.

$$z(z-1)(z-2) = 0, \quad z = 0, 1, 2$$

The function has poles at $z = 0$, $z = 1$ and $z = 2$ of which the given circle encloses the pole at $z = 0$ and $z = 1$.



Residue of $f(z)$ at the simple pole $z = 0$ is

$$= \lim_{z\to 0} z\frac{4-3z}{z(z-1)(z-2)} = \lim_{z\to 0}\frac{4-3z}{(z-1)(z-2)}$$

$$= \frac{4-0}{(0-1)(0-2)} = 2$$

Residue of $f(z)$ at the simple pole $z = 1$ is

$$= \lim_{z\to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)} = \lim_{z\to 1}\frac{4-3z}{z(z-2)} = \frac{4-3}{1(1-2)} = -1$$

By Cauchy's integral formula

$$\int_C f(z)dz = 2\pi i \times \text{ sum of the residue within } c$$
$$= 2\pi i \times (2-1) = 2\pi i$$ **Ans.**

Example 31. *Evaluate*

$$\int_C \frac{12z-7}{(z-1)^2(2z+3)}dz, \text{ where } C \text{ is the circle}$$

(i) $|z| = 2$ (ii) $|z+i| = \sqrt{3}$ (U.P. III Sem. Jan 2011)

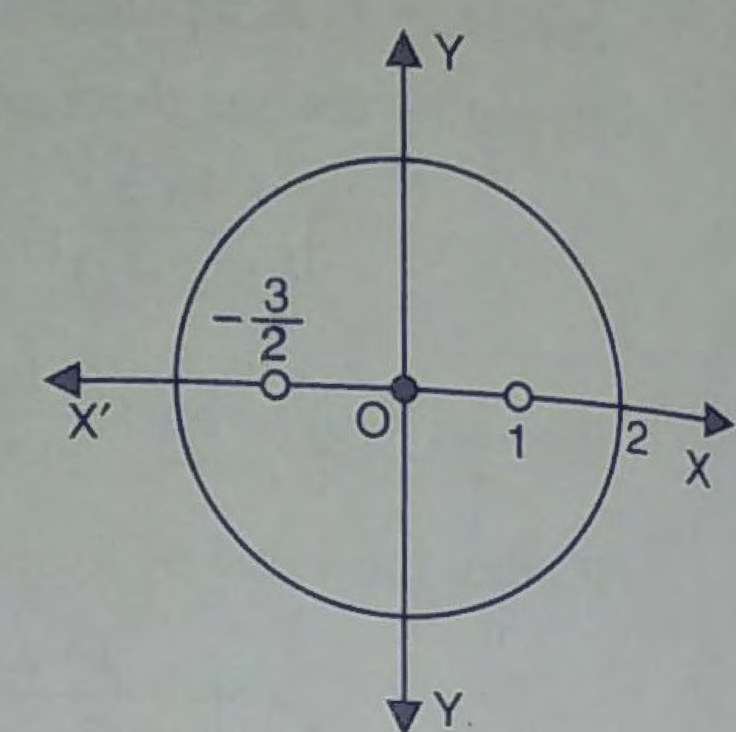
Solution. We have, $f(z) = \dfrac{12z-7}{(z-1)^2(2z+3)}$

Poles are given by

$z = 1$ (double pole) and $z = -\dfrac{3}{2}$ (simple pole)

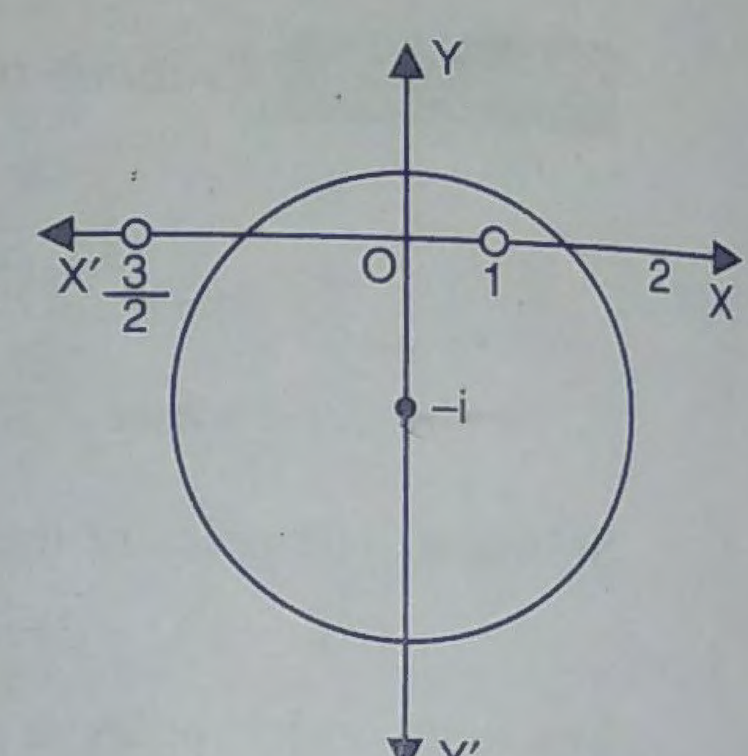
Residue at ($z = 1$) is

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-1)^2 \cdot \frac{12z-7}{(z-1)^2(2z+3)}\right\}\right]_{z=1}$$
$$= \left[\frac{d}{dz}\left(\frac{12z-7}{2z+3}\right)\right]_{z=1} = \left[\frac{(2z+3).12-(12z-7).2}{(2z+3)^2}\right]_{z=1}$$
$$= \frac{60-10}{25} = \frac{50}{25} = 2$$



Residue at simple pole $\left(z = -\dfrac{3}{2}\right)$ is

$$R_2 = \lim_{z\to -3/2}\left(z+\frac{3}{2}\right)\cdot\frac{12z-7}{(z-1)^2(2z+3)}$$
$$= \lim_{z\to -3/2}\frac{1}{2}\cdot\frac{(12z-7)}{(z-1)^2} = -2.$$



(i) The contour $|z| = 2$ encloses both the poles 1 and $-\dfrac{3}{2}$.

∴ The given integral $= 2\pi i\,(R_1 + R_2) = 2\pi i\,(2-2) = 0$.

(ii) The contour $|z+i| = \sqrt{3}$ is a circle of radius $\sqrt{3}$ and centre at $z = -i$. The distances of the centre from $z = 1$ and $-\dfrac{3}{2}$ are respectively $\sqrt{2}$ and $\sqrt{\dfrac{13}{4}}$. The first of these is $< \sqrt{3}$ and the second is $> \sqrt{3}$.

∴ The second contour includes only the first singularity $z = 1$.

Hence, the given integral $= 2\pi i\,(R_1) = 2\pi i\,(2) = 4\pi i$. **Ans.**

Example 32. *Evaluate the complex integral*

$$\int_c \frac{\coth z\, dz}{(z-i)}, \quad c: |z| = 2.$$

Solution. $\displaystyle\int_c \frac{\coth z}{(z-i)}dz = \int_c \frac{e^z+e^{-z}}{(e^z-e^{-z})(z-i)}dz$

The poles of the integrand are given by

$$(e^z - e^{-z})(z-i) = 0$$
$$e^z - e^{-z} = 0 \text{ and } z - i = 0 \text{ i.e. } e^{2z} = 1 \Rightarrow z = 0 \text{ and } z = i$$

i.e. $z = 0$ and $z = i$

Both the poles are inside $c : |z| = 2$.

$$\text{Residue (at } z = i) = \lim_{z\to i}(z-i)\frac{e^z+e^{-z}}{(e^z-e^{-z})(z-i)} = \frac{e^i+e^{-i}}{e^i-e^{-i}} = \coth i$$

To find the residue at $z = 0$, we apply $\dfrac{\phi(z)}{\psi(z)}$ method

$$\frac{\phi(z)}{\psi(z)} = \frac{\frac{e^z+e^{-z}}{z-i}}{e^z-e^{-z}}, \quad \frac{\phi(z)}{\psi'(z)} = \frac{\frac{e^z+e^{-z}}{z-i}}{e^z+e^{-z}}$$

$$\text{Residue [at } (z=0)] = \left[\frac{\phi(0)}{\psi'(0)}\right] = \frac{\frac{1+1}{0-i}}{1+1} = \frac{-1}{i} = i$$

Sum of the residues = $\coth i + i$

By Cauchy's Residue Theorem

$$\int_c f(z)\,dz = 2\pi i \times \text{Sum of the residues}$$
$$\int_c \frac{\coth z}{z-i}dz = 2\pi i[\coth i + i]$$ **Ans.**

Example 33. *Determine the poles of the following function and residue at each pole:*

$$f(z) = \frac{z^2}{(z-1)^2(z+2)} \text{ and hence evaluate}$$

$$\int_c \frac{z^2 dz}{(z-1)^2(z+2)} \text{ where } c: |z| = 3. \text{ (R.G.P.V. Bhopal, III Sem. Dec. 2007)}$$

Solution. $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

Poles of $f(z)$ are given by $(z-1)^2(z+2) = 0$ i.e. $z = 1, 1, -2$

The pole at $z = 1$ is of second order and the pole at $z = -2$ is simple.

$$\text{Residue of } f(z) \text{ (at } z = 1) = \lim_{z\to 1}\frac{1}{(2-1)!}\frac{d}{dz}\frac{(z-1)^2 z^2}{(z-1)^2(z+2)}$$
$$= \lim_{z\to 1}\frac{d}{dz}\frac{z^2}{z+2} = \lim_{z\to 1}\frac{(z+2)2z - 1.z^2}{(z+2)^2}$$
$$= \lim_{z\to 1}\frac{z^2+4z}{(z+2)^2} = \frac{1+4}{(1+2)^2} = \frac{5}{9}$$

Residue of $f(z)$ (at $z = -2$) $= \lim_{z \to -2} \frac{(z+2)z^2}{(z-1)^2(z+2)} = \lim_{z \to -2} \frac{z^2}{(z-1)^2}$

$$= \frac{4}{(-2-1)^2} = \frac{4}{9}$$

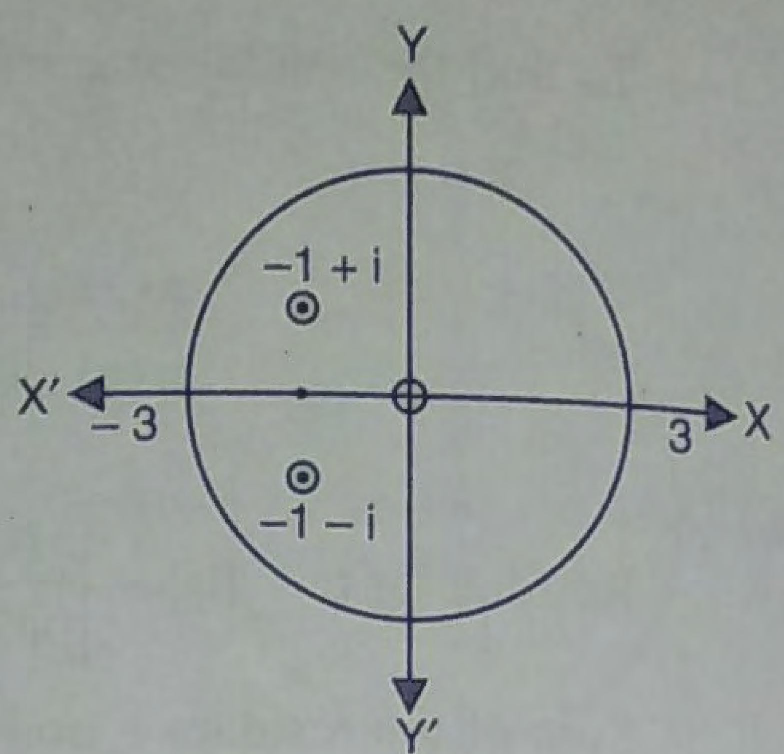
$$\int_c \frac{z^2\,dz}{(z-1)^2(z+2)} = 2\pi i\left(\frac{5}{9}+\frac{4}{9}\right) = 2\pi i$$ **Ans.**

Example 34. *Using Residue theorem, evaluate* $\frac{1}{2\pi i}\int_C \frac{e^{zt}\,dz}{z^2(z^2+2z+2)}$

where C is the circle $|z| = 3$. (U.P., III Semester, Dec. 2009)

Solution. Here, we have

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}\,dz}{z^2(z^2+2z+2)}$$



Poles are given by

$z = 0$ (double pole)

$z = -1 \pm i$ (simple poles)

$\Rightarrow z^2 + 2z + 2 = 0$

$\Rightarrow z^2 + 2z + 1 = -1$

$\Rightarrow (z+1)^2 = -1$

$\Rightarrow z + 1 = \pm i$

$\Rightarrow z = -1 \pm i$

All the four poles are inside the given circle $|z| = 3$

Residue (at $z = 0$) is $\lim_{z \to 0} \frac{d}{dz} z^2 \frac{e^{zt}}{z^2(z^2+2z+2)}$

$$= \lim_{z \to 0} \frac{d}{dz} \frac{e^{zt}}{z^2+2z+2}$$

$$= \lim_{z \to 0} \frac{(z^2+2z+2)\,t e^{zt} - (2z+2)\,e^{zt}}{(z^2+2z+2)^2}$$

$$= \frac{2te^0 - 2e^0}{4} = \frac{(t-1)}{2}$$

Residue (at $z = -1 + i$) $= \lim_{z \to -1+i} \frac{(z+1-i)\,e^{zt}}{z^2(z+1-i)(z+1+i)}$

$$= \lim_{z \to -1+i} \frac{e^{zt}}{z^2(z+1+i)} = \frac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)}$$

$$= \frac{e^{(-1+i)t}}{(1-2i-1)(2i)} = \frac{e^{(-1+i)t}}{4}$$

Similarly, Residue at $(z = -1 - i) = \frac{e^{(-1-i)t}}{4}$

$$\int \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = 2\pi i \text{ (Sum of the Residues)}$$

$$\Rightarrow \frac{1}{2\pi i}\int \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}(e^{it} + e^{-it})$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}(2\cos t)$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\cos t$$ **Ans.**

Example 35. *Evaluate* $\oint_C \frac{1}{\sinh z}\,dz$, *where C is the circle* $|z| = 4$.

Solution. Here, $f(z) = \frac{1}{\sinh z}$.

Poles are given by

$\sinh z = 0$

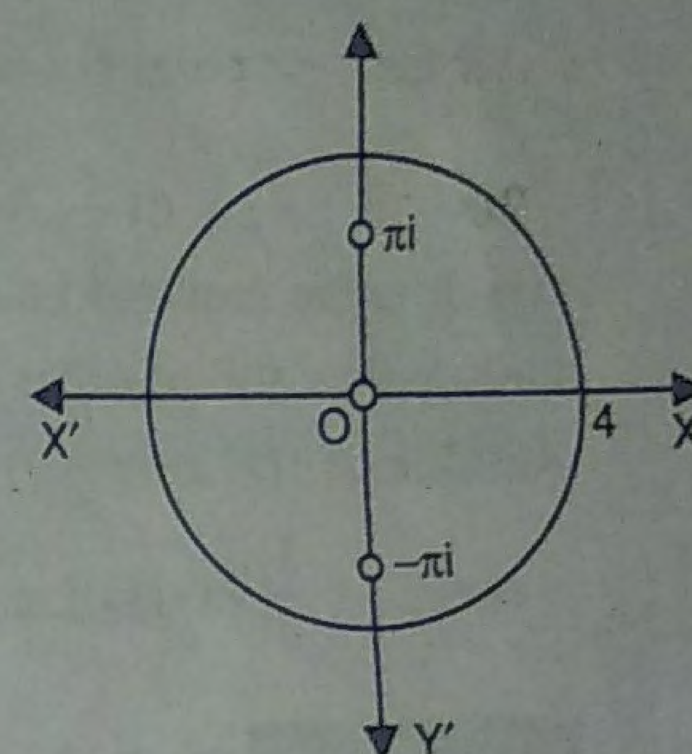
$\sin iz = 0$

$\Rightarrow z = n\pi i$ where n is an integer.

$\Rightarrow$ Out of these, the poles $z = -\pi i$, 0 and πi lie inside the circle $|z| = 4$.

The given function $\frac{1}{\sinh z}$ is of the form $\frac{\phi(z)}{\psi(z)}$

Its pole at $z = a$ is $\frac{\phi(a)}{\psi'(a)}$.



Residue (at $z = -\pi i$)

$$= \frac{1}{\cosh(-\pi i)} = \frac{1}{\cos i(-\pi i)} = \frac{1}{\cos \pi}$$

$$= \frac{1}{-1} = -1$$

Residue (at $z = 0$) $= \frac{1}{\cosh 0} = \frac{1}{1} = 1$

Residue (at $z = \pi i$) $= \frac{1}{\cosh(\pi i)} = \frac{1}{\cos i(\pi i)} = \frac{1}{\cos(-\pi)}$

$$= \frac{1}{\cos \pi} = \frac{1}{-1} = -1$$

Residues at $-\pi i$, 0, πi are respectively -1, 1 and -1.

Hence, the required integral $= 2\pi i(-1 + 1 - 1) = -2\pi i$. **Ans.**

Example 36. *Obtain Laurent's expansion for the function* $f(z) = \frac{1}{z^2 \sinh z}$ *at the isolated singularity and hence evaluate* $\oint_C \frac{1}{z^2 \sinh z}\,dz$, *where C is the circle* $|z - 1| = 2$.

Solution. Here, $f(z) = \frac{1}{z^2 \sinh z} = \frac{2}{z^2(e^z - e^{-z})}$

$$= \frac{2}{z^2\left[\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\right)-\left(1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\ldots\right)\right]}$$

$$= \frac{2}{z^2\left(2z+\frac{2z^3}{3!}+\frac{2z^5}{5!}+\ldots\right)}$$

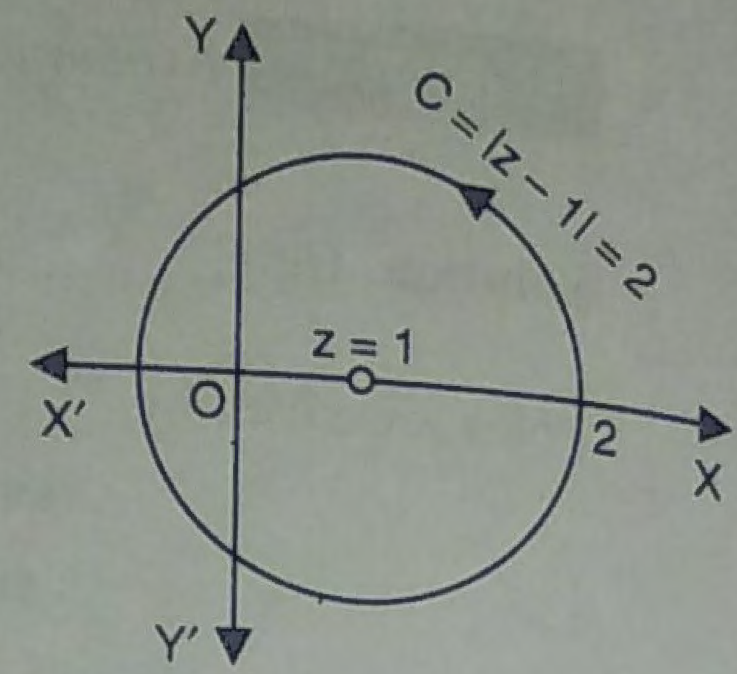
$$= \frac{1}{z^3\left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+\ldots\right)}$$

$$= z^{-3}\left[1+\left(\frac{z^2}{6}+\frac{z^4}{120}\right)+\ldots\right]^{-1}$$

$$= z^{-3}\left(1-\frac{z^2}{6}-\frac{z^4}{120}+\ldots\right)$$

$$= z^{-3}-\frac{z^{-1}}{6}-\frac{z}{120}+\ldots$$

$$f(z) = \frac{1}{z^3}-\frac{1}{6z}-\frac{z}{120}+\ldots$$



which is the required Laurent's expansion.

Only pole $z = 0$ of order three lies inside the circle $c = |z-1| = 2$.

Residue of $f(z)$ at $(z = 0)$ is

$= \text{coeff. of } \frac{1}{z}$ in the Laurent's expansion of $f(z) = -\frac{1}{6}$ **Ans.**

Example 37. *Evaluate the complex integral* $\int_c \frac{dz}{\cosh(z)}$ *where c is* $|z| = 2$.

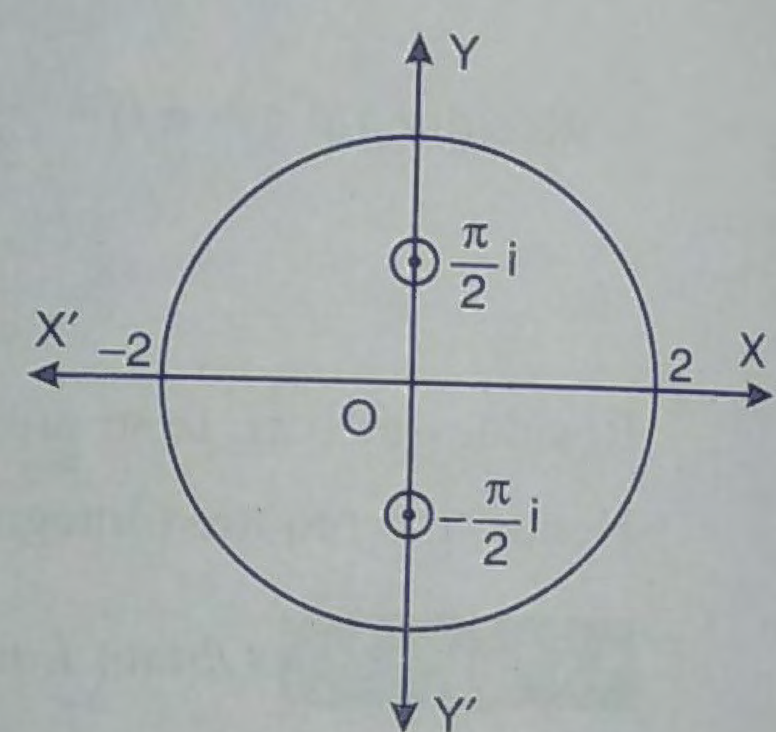
Solution. $f(z) = \frac{1}{\cosh(z)} = \frac{2}{e^z+e^{-z}} = \frac{2e^z}{e^{2z}+1}$

Poles of $f(z)$ are given by

$$e^{2z}+1 = 0 \Rightarrow (e^z+i)(e^z-i) = 0 \Rightarrow e^z = i, -i$$

$$\Rightarrow \quad e^z = e^{\frac{\pi}{2}i}, \quad e^z = e^{-\frac{\pi}{2}i}$$

$$\Rightarrow \quad z = \frac{\pi}{2}i, \quad z = -\frac{\pi}{2}i$$



The poles which lie within the contour $|z| = 2$ are

$$z = \frac{\pi}{2}i \text{ and } \frac{-\pi i}{2}$$

Residue of $f(z)$ $\left(\text{at } z = \frac{\pi i}{2}\right) = \left\{\frac{2e^z}{\frac{d}{dz}(e^{2z}+1)}\right\}_{z=\frac{\pi}{2}i}$ $\left[\text{Residue} = \frac{\phi(a)}{\psi'(a)}\right]$

$$\left\{\frac{\ldots}{2e^{2z}}\right\}_{z=\frac{\pi}{2}i} = \{e^{-z}\}_{z=\frac{\pi}{2}i} = e^{-\frac{\pi}{2}i} = -i$$

Residue of $f(z)\left(\text{at } z = -\frac{\pi}{2}i\right) = \left\{\frac{2e^z}{\frac{d}{dz}(e^{2z}+1)}\right\}_{z=-\frac{\pi}{2}i} = \left\{\frac{2e^z}{2e^{2z}}\right\} = \{e^{-z}\}_{z=-\frac{\pi}{2}i} = e^{\frac{\pi i}{2}} = i$

$$\int_c f(z)\,dz = \int_c \frac{dz}{\cosh z} = 2\pi i \text{ [sum of residues]} = 2\pi i\,(i-i) = 0 \quad \textbf{Ans.}$$

Example 38. *Evaluate* $\int_c \frac{dz}{z\sin z}$ *: c is the unit circle about origin.*

Solution. $\frac{1}{z\sin z} = \frac{1}{z\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\right]} = \frac{1}{z^2\left[1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots\right]}$

$$= \frac{1}{z^2}\left[1-\left(\frac{z^2}{6}-\frac{z^4}{120}\ldots\right)\right]^{-1} = \frac{1}{z^2}\left[1+\left(\frac{z^2}{6}-\frac{z^4}{120}\right)+\left(\frac{z^2}{6}-\frac{z^4}{120}\ldots\right)^2\ldots\right]$$

$$= \frac{1}{z^2}\left[1+\frac{z^2}{6}-\frac{z^4}{120}+\frac{z^4}{36}+\ldots\right] = \frac{1}{z^2}+\frac{1}{6}-\frac{z^2}{120}+\frac{z^4}{36}\ldots$$

$$= \frac{1}{z^2}+\frac{1}{6}+\frac{7}{360}z^2\ldots$$

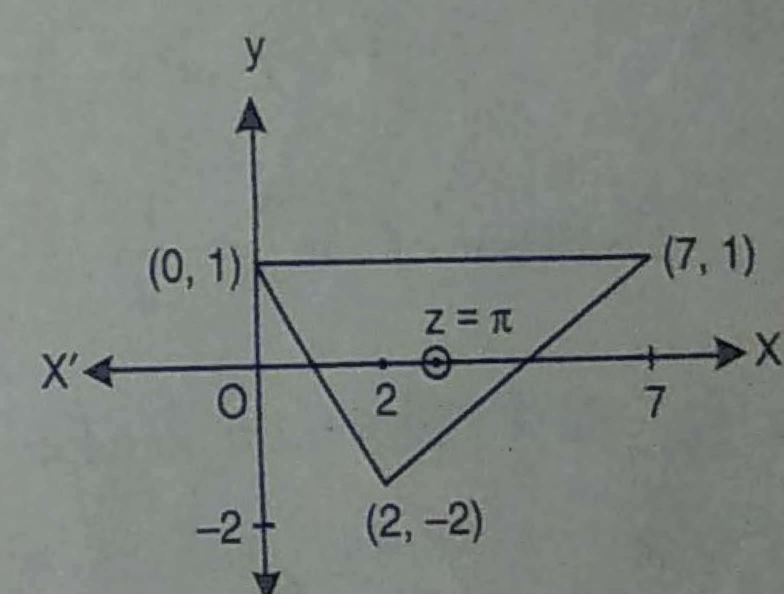
This shows that $z = 0$ is a pole of order 2 for the function $\frac{1}{z\sin z}$ and the residue at the pole is zero, (coefficient of $\frac{1}{z}$).

Now the pole at $z = 0$ lies within C.

$$\therefore \quad \int_C \frac{1}{z\sin z}\,dz = 2\pi i \text{ (Sum of Residues)} = 0 \quad \textbf{Ans.}$$

Example 39. *Evaluate* $\oint_C \frac{1}{z^2\sin z}\,dz$ *where, C is triangle with vertices (0, 1), (2, –2), (7, 1).*

Soluiton. Here, $f(z) = \frac{1}{z^2\sin z}$.



Poles are given by $z^2\sin z = 0$

$\Rightarrow$ $z = 0$ (double pole)

and $z = n\pi$ $(n = 1, 2, 3, \ldots)$

Only $z = \pi$ lies inside contour C.

Residue (at $z = \pi$) is

15. Residue at $z = 0$ of the function $f(z) = z^2 \sin\frac{1}{z}$ is

(i) $\frac{1}{6}$ (ii) $-\frac{1}{6}$ (iii) $\frac{2}{3}$ (iv) $-\frac{2}{3}$ **Ans.** (ii)

16. The residue at the poles of the function $f(z) = \cot z$, equals,

(i) 0 (ii) 1 (iii) –1 (iv) $2\pi i$ **Ans.** (ii)

17. The function $(z-1)\sin\frac{1}{z}$ at $z = 0$ has

(i) a removable singularity (ii) a simple pole

(iii) an essential singularity (iv) a multile pole. **Ans.** (iii)

15.13 EVALUATION OF REAL DEFINITE INTEGRALS BY CONTOUR INTEGRATION

A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using Cauchy's theorem of residues. For finding the integrals we take a closed curve C, find the poles of the function $f(z)$ and calculate residues at those poles only which lie within the curve C.

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of the residues of } f(z) \text{ at the poles within } C)$$

We call the curve, a contour and the process of integration along a contour is called contour integration.

15.14 INTEGRATION ROUND THE UNIT CIRCLE OF THE TYPE

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$$

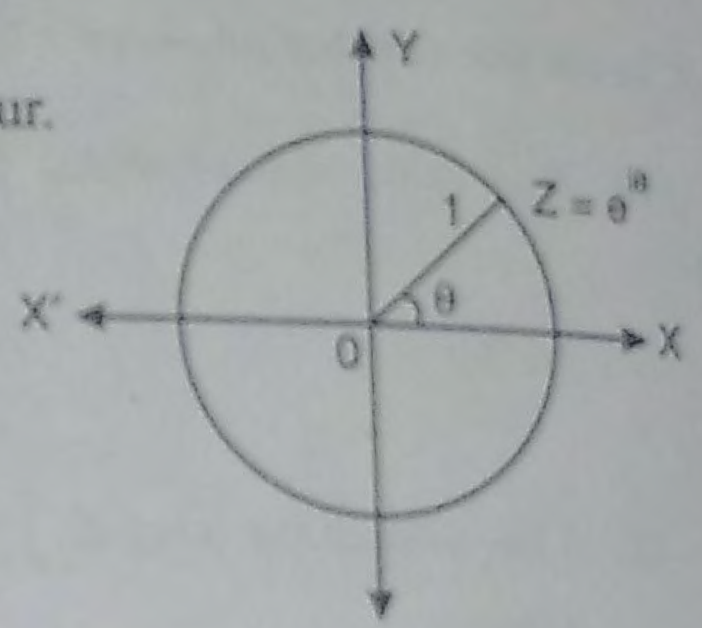
where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Convert $\sin\theta$, $\cos\theta$ into z.

Consider a circle of unit radius with centre at origin, as contour.

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left[z - \frac{1}{z}\right], \qquad z = re^{i\theta} = 1.e^{i\theta} = e^{i\theta}$$

$$\cos\theta = \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left[z + \frac{1}{z}\right]$$



As we know

$$z = e^{i\theta},\ dz = e^{i\theta}\,i\,d\theta = zi\,d\theta \text{ or } d\theta = \frac{dz}{iz}$$

The integrand is converted into a function of z.

Then apply Cauchy's residue theorem to evaluate the integral.

Some examples of these are illustrated below.

Example 42. *Evaluate the integral:*

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$$ *(R.G.P.V., Bhopal, III Semester, June 2007)*

Solution. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5 - 3\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$

$$= \int_0^{2\pi} \frac{2\,d\theta}{10 - 3e^{i\theta} - 3e^{-i\theta}}$$

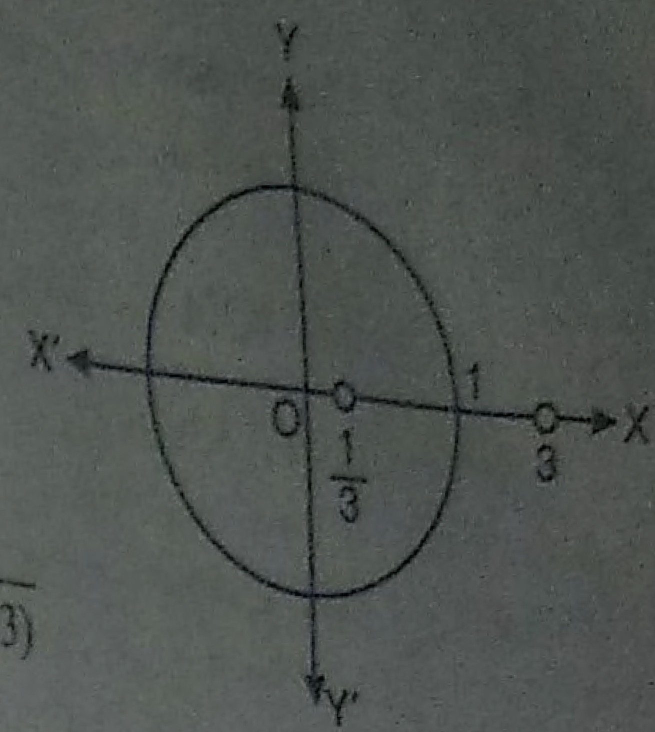
$$\left[e^{i\theta} = z \Rightarrow i.e^{i\theta} d\theta = dz, \quad d\theta = \frac{dz}{iz}\right]$$

$$= \int_C \frac{2}{10 - 3z - \frac{3}{z}} \frac{dz}{iz} = \frac{2}{i}\int_C \frac{dz}{(3z-1)(z-3)}$$

[C is the unit circle $|z| = 1$]

$$= -\frac{2}{i}\int_C \frac{dz}{3z^2 - 10z + 3}$$

$$= -\frac{2}{i}\int_C \frac{dz}{(3z-1)(z-3)} = 2i\int_C \frac{dz}{(3z-1)(z-3)}$$



Let $\displaystyle I = 2i\int_C \frac{dz}{(3z-1)(z-3)}$

Poles of the integrand are given by

$$(3z-1)(z-3) = 0 \Rightarrow z = \frac{1}{3}, 3$$

There is only one pole at $z = \frac{1}{3}$ inside the unit circle C.

Residue at $z = \frac{1}{3}$ $\displaystyle = \lim_{z\to\frac{1}{3}} \left(z - \frac{1}{3}\right) f(z) = \lim_{z\to\frac{1}{3}} \frac{2i\left(z-\frac{1}{3}\right)}{(3z-1)(z-3)} = \lim_{z\to\frac{1}{3}} \frac{2i}{3(z-3)}$

$$= \frac{2i}{3\left(\frac{1}{3} - 3\right)} = -\frac{i}{4}$$

Hence, by Cauchy's Residue Theorem

$$I = 2\pi i \text{ (Sum of the residues within Contour)} = 2\pi i\left(\frac{-i}{4}\right) = \frac{\pi}{2}$$

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = \frac{\pi}{2}$$ **Ans.**

Example 43. *Use residue calculus to evaluate the following integral*

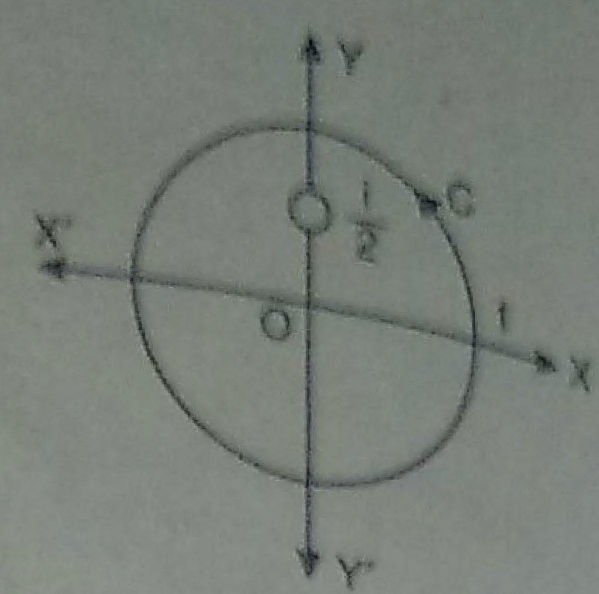
$$\int_0^{2\pi} \frac{1}{5 - 4\sin\theta}\,d\theta$$

Solution. Let $\displaystyle I = \int_0^{2\pi} \frac{1}{5 - 4\sin\theta}\,d\theta = \int_0^{2\pi} \frac{1}{5 - 4\frac{e^{i\theta} - e^{-i\theta}}{2i}}\,d\theta$

$$= \int_0^{2\pi} \frac{d\theta}{5 + 2ie^{i\theta} - 2ie^{-i\theta}}$$ [putting $e^{i\theta} = z,\ d\theta = \frac{dz}{iz}$]

$$= \int_c \frac{1}{5 + 2iz - \frac{2i}{z}} \frac{dz}{iz}$$ where c is the unit circle $|z| = 1$.

$$= \int_c \frac{dz}{5iz - 2z^2 + 2}$$



Poles of integrand are given by

$$-2z^2 + 5iz + 2 = 0 = 0 \Rightarrow z = \frac{-5i \pm \sqrt{-25+16}}{-4} = \frac{-5i \pm 3i}{-4} = 2i, \frac{i}{2}$$

Only $z = \frac{i}{2}$ lies inside c.

Residue at the simple pole at $z = \frac{i}{2}$ is

$$\lim_{z \to \frac{i}{2}} \left(z - \frac{i}{2}\right) \times \left[\frac{1}{(2z - i)(-z + 2i)}\right] = \lim_{z \to \frac{i}{2}} \frac{1}{2(-z + 2i)} = \frac{1}{2\left(-\frac{i}{2} + 2i\right)} = \frac{1}{3i}$$

Hence, by Cauchy's residue theorem

$$I = 2\pi i \times \text{Sum of residues within the contour}$$

$$= 2\pi i \times \frac{1}{3i} = \frac{2\pi}{3}$$

Hence, given integral $= \frac{2\pi}{3}$ **Ans.**

Example 44. *Evaluate* $\int_0^{2\pi} \frac{d\theta}{a + b \sin\theta}$ if $a > |b|$

(G.B.T.U., 2012, U.P. III Semester 2009-2010)

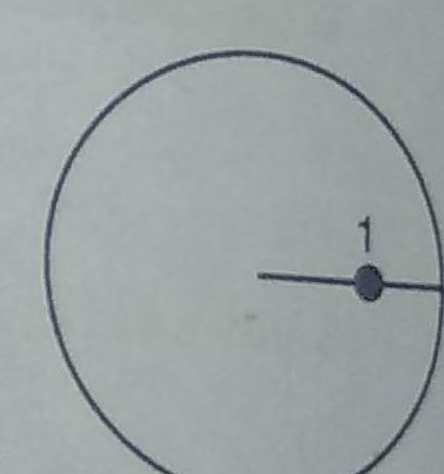
Solution. Let $$I = \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$$

$$= \int_0^{2\pi} \frac{1}{a + b\frac{e^{i\theta} - e^{-i\theta}}{2i}} d\theta$$ $\left[e^{i\theta} = z, d\theta = \frac{dz}{iz}\right]$

$$= \int_C \frac{1}{a + \frac{b}{2i}\left(z - \frac{1}{z}\right)} \frac{dz}{iz}$$ (where C is the unit circle $|z| = 1$)

$$= \int_c \frac{2}{2iaz + bz^2 - b} dz$$

$$= \int_c \frac{2}{bz^2 + 2aiz - b} dz = \frac{1}{b} \int \frac{2\,dz}{z^2 + \frac{2aiz}{b} - 1}$$



$$bz^2 + 2aiz - b = b\left[z^2 + \frac{2aiz}{b} - 1\right]$$

Where $$\alpha + \beta = -\frac{2ai}{b}$$

$$\alpha\beta = -1$$

$\left[(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = -\frac{4a^2}{b^2} + 4,\quad \alpha - \beta = 2\frac{\sqrt{b^2 - a^2}}{b}\right]$

$|\alpha| < 1$ then $|\beta| > 1$

i.e.: Pole lies at $z = \alpha$ in the unit circle.

$$\text{Residue at } z = \alpha = \lim_{z \to \alpha} (z - \alpha) \frac{2}{(z - \alpha)(z - \beta)} = \frac{2}{\alpha - \beta} = \frac{b}{\sqrt{b^2 - a^2}} = \frac{b}{i\sqrt{a^2 - b^2}}$$

$$\int_0^{2\pi} \frac{1}{a + b\sin\theta} d\theta = \frac{1}{b} \int_c \frac{2}{z^2 + 2\frac{aiz}{b} - 1} dz = 2\pi i \frac{b}{bi\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$ **Ans.**

Example 45. *Evaluate* $\int_0^{\pi} \frac{d\theta}{3 + 2\cos\theta}$ *by contour integration in the complex plane.*

Solution. $$\int_0^{\pi} \frac{d\theta}{3 + 2\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3 + (e^{i\theta} + e^{-i\theta})}$$ Putting $e^{i\theta} = z,\ d\theta = \frac{dz}{iz}$

$$= \frac{1}{2}\int_c \frac{\frac{dz}{iz}}{3 + z + \frac{1}{z}} = \frac{1}{2i}\int_C \frac{dz}{z^2 + 3z + 1}$$, where c is the unit circle $|z| = 1$.

Poles are given by $z^2 + 3z + 1 = 0$ or $z = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$

There are two poles at $z = \frac{-3 + \sqrt{5}}{2}$ and $z = \frac{-3 - \sqrt{5}}{2}$

Only one of these poles at $z = \frac{-3 + \sqrt{5}}{2}$ is inside the circle.

Residue at $\left(z = \frac{-3 + \sqrt{5}}{2}\right)$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3 + \sqrt{5}}{2}\right) \frac{1}{\left(z - \frac{-3 + \sqrt{5}}{2}\right)\left(z - \frac{-3 - \sqrt{5}}{2}\right)} = \frac{1}{\frac{-3 + \sqrt{5}}{2} - \frac{-3 - \sqrt{5}}{2}} = \frac{1}{\sqrt{5}}$$

Hence by Cauchy Residue theorem

$$\frac{1}{2i}\int_c \frac{dz}{z^2 + 3z + 1} = \frac{1}{2i}\left[2\pi i \times \text{Residue at}\left(z = \frac{-3 + \sqrt{5}}{2}\right)\right] = \frac{1}{2i} \times 2\pi i \times \frac{1}{\sqrt{5}} = \frac{\pi}{\sqrt{5}}$$

$$\int_0^{\pi} \frac{d\theta}{3 + 2\cos\theta} = \frac{\pi}{\sqrt{5}}$$ **Ans.**

$\int_0 \frac{\quad}{2+\cos\theta}$.

Solution. Let $I=\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=\int_0^{2\pi}\frac{d\theta}{2+\frac{e^{i\theta}+e^{-i\theta}}{2}}=\int_0^{2\pi}\frac{2d\theta}{4+e^{i\theta}+e^{-i\theta}}$

Put $e^{i\theta}=z$ so that $e^{i\theta}(i\,d\theta)=dz \Rightarrow iz\,d\theta=dz \Rightarrow d\theta=\frac{dz}{iz}$

$$I=\int_C\frac{2\frac{dz}{iz}}{4+z+\frac{1}{z}}$$ where c denotes the unit circle $|z|=1$.

$$=\frac{1}{i}\int_c\frac{2\,dz}{z^2+4z+1}$$

The poles are given by putting the denominator equal to zero.

$$z^2+4z+1=0 \text{ or } z=\frac{-4\pm\sqrt{16-4}}{2}=\frac{-4\pm2\sqrt{3}}{2}=-2\pm\sqrt{3}$$

The pole within the unit circle C is a simple pole at $z=-2+\sqrt{3}$. Now we calculate the residue at this pole.

$$\text{Residue at } (z=-2+\sqrt{3})=\lim_{z\to(-2+\sqrt{3})}\frac{1}{i}\,\frac{(z+2-\sqrt{3})2}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

$$=\lim_{z\to(-2+\sqrt{3})}\frac{2}{i(z+2+\sqrt{3})}=\frac{2}{i(-2+\sqrt{3}+2+\sqrt{3})}=\frac{1}{\sqrt{3}i}$$

Hence by Cauchy's Residue Theorem, we have

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=2\pi i \text{ (sum of the residues within the contour)}$$

$$=2\pi i\,\frac{1}{i\sqrt{3}}=\frac{2\pi}{\sqrt{3}}$$ **Ans.**

Example 47. *Using complex variable techniques evaluate the real integral*

$$\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{5-4\cos\theta}$$

Solution. If we put $z=e^{i\theta}$

$$\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right),\quad d\theta=\frac{dz}{iz}$$

and so $$I=\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}d\theta=\frac{1}{2}\int_0^{2\pi}\frac{1-\cos2\theta}{5-4\cos\theta}d\theta$$ [where c is a circle of unit radius with centre z = 0]

$$=\text{Real part of }\frac{1}{2}\int_0^{2\pi}\frac{1-e^{2i\theta}}{5-4\cos\theta}d\theta$$

$$=\text{Real part of }\frac{1}{2}\int_c\frac{1-z^2}{5-2\left(z+\frac{1}{z}\right)}\left(\frac{dz}{iz}\right)$$

$$=\text{Real part of }\frac{1}{2i}\int_c\frac{1-z^2}{5z-2z^2-2}dz$$

$$=\text{Real part of }\frac{1}{2i}\int_c\frac{z^2-1}{2z^2-5z+2}dz$$

Poles are determined by $2z^2-5z+2=0$ or $(2z-1)(z-2)=0$ or $z=\frac{1}{2},2$

So inside the contour c there is a simple pole at $z=\frac{1}{2}$

$$\text{Residue at the simple pole }\left(z=\frac{1}{2}\right)=\lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^2-1}{(2z-1)(z-2)}$$

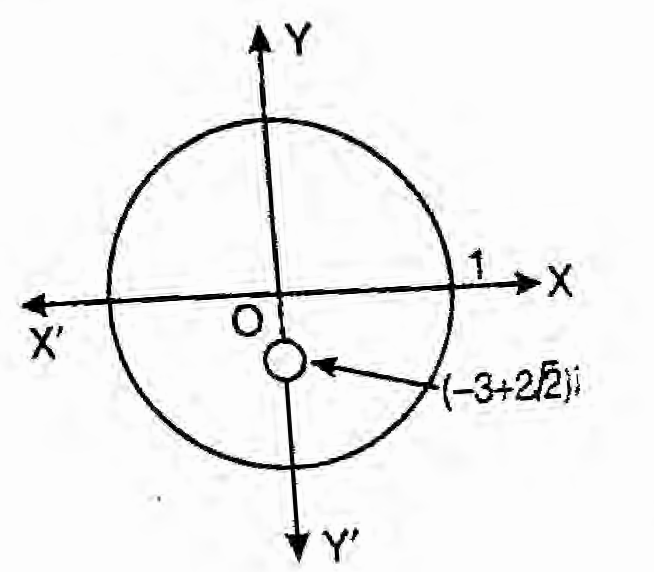
$$=\lim_{z\to\frac{1}{2}}\frac{z^2-1}{2(z-2)}=\frac{\frac{1}{4}-1}{2\left(\frac{1}{2}-2\right)}=\frac{1}{4}$$

$$I=\text{Real part of }\frac{1}{2i}\int_c\frac{(z^2-1)}{2z^2-5z+2}dz=\frac{1}{2i}2\pi i\text{ (sum of the residues)}$$

$$\Rightarrow\quad\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}d\theta=\pi\left(\frac{1}{4}\right)=\frac{\pi}{4}$$ **Ans.**

Example 48. *Using the complex variable techniques, evaluate the real integral*

$$\int_0^{2\pi}\frac{\cos\theta}{3+\sin\theta}d\theta.$$



Solution. Let $I=\int_0^{2\pi}\frac{\cos\theta}{3+\sin\theta}d\theta$

$$I=\text{Real part of }\int_0^{2\pi}\frac{e^{i\theta}}{3+\sin\theta}d\theta$$

$$=\text{Real part of }\int_0^{2\pi}\frac{e^{i\theta}}{3+\frac{e^{i\theta}-e^{-i\theta}}{2i}}d\theta$$

Putting $e^{i\theta}=z$ so that $e^{i\theta}\,i\,d\theta=dz \Rightarrow iz\,d\theta=dz$ or $d\theta=\frac{dz}{iz}$

$$= \frac{2ai}{(\beta-\alpha)} = \frac{2ai}{(2a^2+1-2a\sqrt{a^2+1})-(2a^2+1+2a\sqrt{a^2+1})}$$

$$= \frac{2ai}{-4a\sqrt{a^2+1}} = -\frac{i}{2\sqrt{a^2+1}}$$

· Cauchy's residue theorem

I = $2\pi i$ (sum of the residues within the contour c)

$$= 2\pi i\frac{-i}{2\sqrt{a^2+1}} = \frac{\pi}{\sqrt{a^2+1}}$$

$$\int^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$$ **Proved.**

Evaluate by Contour integration:

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta .$$

Let $$I = \int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta-n\theta)+i\sin(\sin\theta-n\theta)]\,d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta}e^{i(\sin\theta-n\theta)}d\theta = \int_0^{2\pi} e^{\cos\theta+i\sin\theta}\cdot e^{-ni\theta}d\theta$$

$$= \int_0^{2\pi} e^{e^{i\theta}}\cdot e^{-in\theta}d\theta \qquad ...(1)$$

that $d\theta = \frac{dz}{iz}$ then,

$$I = \int_C e^z\cdot\frac{1}{z^n}\cdot\frac{dz}{iz} = -i\int_C\frac{e^z}{z^{n+1}}dz$$

of order $(n+1)$.

e unit circle.

at $z = 0$ is

$$= \frac{1}{(n+1-1)!}\left[\frac{d^n}{dz^n}\left\{z^{n+1}\cdot\frac{-ie^z}{z^{n+1}}\right\}\right]_{z=0}$$

$$= \frac{-i}{n!}\left[\frac{d^n}{dz^n}(e^z)\right]_{z=0} = \frac{-i}{n!}(e^z)_{z=0} = \frac{-i}{n!}$$

Residue theorem,

$$I = 2\pi i\left(\frac{-i}{n!}\right) = \frac{2\pi}{n!}$$

of $$\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta-n\theta)+i\sin(\sin\theta-n\theta)]\,d\theta = \frac{2\pi}{n!},$$ **Ans.**

2π

EXERCISE 15.4

Evaluate the following integrals:

1. $\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta$ (*R.G.P.V., Bhopal, III Semester, June 2008*) **Ans.** [illegible]
2. $\int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos n\theta}{3+2\cos\theta}d\theta$ **Ans.** $\frac{2\pi}{\sqrt5}(3-\sqrt5)^n, n>0$
3. $\int_0^{2\pi}\frac{4}{5+4\sin\theta}d\theta$ **Ans.** $\frac{8\pi}{3}$
4. $\int_0^{\pi}\frac{d\theta}{17-8\cos\theta}$ **Ans.** [illegible]
5. $\int_0^{\pi}\frac{d\theta}{a+b\cos\theta}$, *where* $a > |b|$. *Hence or otherwise evaluate* $\int_0^{2\pi}\frac{d\theta}{\sqrt2-\cos\theta}$. **Ans.** $\frac{\pi}{\sqrt{a^2 ...}}$ [illegible]

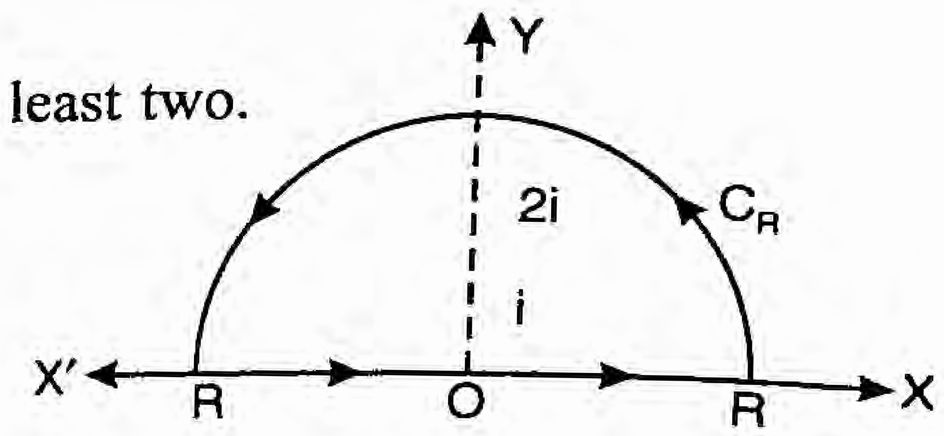
15.15 EVALUATION OF $\int_{-\infty}^{\infty}\frac{f_1(x)}{f_2(x)}dx$ where $f_1(x)$ and $f_2(x)$ are polynomials in x.

Such integrals can be reduced to contour integrals, if

(*i*) $f_2(x)$ has no real roots.

(*ii*) the degree of $f_2(x)$ is greater than that of $f_1(x)$ by at least two.

Procedure: Let $f(x) = \frac{f_1(x)}{f_2(x)}$



Consider $\int_C f(z)\,dz$

where C is a curve, consisting of the upper half C_R of the circle $|z| = R$, and part of the real axis from $-R$ to R.

If there are no poles of $f(z)$ on the real axis, the circle $|z| = R$ which is arbitrary can be taken such that there is no singularity on its circumference C_R in the upper half of the plane, but possibly some poles inside the contour C specified above.

Using Cauchy's theorem of residues, we have

$$\int_C f(z)\,dz = 2\pi i \times \text{(sum of the residues of } f(z) \text{ at the poles within } C)$$

i.e. $$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \text{ (sum of residues within } C)$$

$$\Rightarrow \quad \int_{-R}^{R} f(x)\,dx = -\int_{C_R} f(z)\,dz + 2\pi i \text{ (sum of residues within } C)$$

$$\therefore \quad \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = -\lim_{R\to\infty}\int_{C_R} f(z)\,dz + 2\pi i \text{ (sum of residues within } C) \quad ...(1)$$

Now, $$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = \int_0^{\pi} f(Re^{i\theta})Rie^{i\theta}d\theta$$
$$= 0$$

(1) reduces $$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ (sum of residues within } C)$$

Example *Evaluate* $\int_0^{\infty}\frac{\cos mx}{(x^2+1)}dx$.

(*U.P., III Sem. Jan 2011, R.G.P... ...l, III Sem.,*

Solution. $\int_0^{\infty}\frac{\cos mx}{x^2+1}dx$